Thoughts on Density Points

03E20, 28A05

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46th Winterschool, Hejnice,
Monday, 29th of January 2018, 16:00–16:20

Definition

Let $S \subset \mathbb{R}$. We say that x is a density point of S if

$$\liminf_{n<\omega} \frac{\lambda(S \cap U_{x,1/n})}{\lambda(U_{x,1/n})} = 1.$$

Theorem (Henri Lebesgue (1904))

Let $S \subset \mathbb{R}$ be Lebesgue-measurable. Then almost all points of S are density points of X.

Definition

1. Suppose that A is a Borel subset of ${}^{\omega}2$. Suppose that $x \in {}^{\omega}2$. Then x is a *weak* $\mathbb{B}$*-density point* of A if for every $T \in \mathbb{B}$ and $s = \mathrm{stem}_T$, there is some n_0 such that for all $n \geq n_0$,

$$f_{x \upharpoonright n} \circ f_s^{-1}[T] \cap A \notin I_{\mathbb{B}}^{*}.$$

2. Let $D_{\mathbb{B}}^{\mathrm{weak}}(A)$ denote the set of weak $\mathbb{B}$-density points of A.
3. We say that $\mathbb{B}$ has the *weak density property* if for every Borel subset A of ${}^{\omega}2$, $A \triangle D_{\mathbb{B}}^{\mathrm{weak}}(A) \in I_{\mathbb{B}}^{*}$.

Lemma

Suppose that A is a Borel subset of ${}^\omega 2$, where as before $\mathbb{B}$ denotes random forcing.

1. *If $\liminf_n \mu_n(x, A) = 1$, then x is a weak $\mathbb{B}$-density point of A.*
2. *If $\liminf_n \mu_n(x, A) = 0$, then x is not a weak $\mathbb{B}$-density point of A.*

Corollary

For every Borel subset A of ${}^\omega 2$,

$$D_L(A) =_{I^*_{\mathbb{B}}} D^{\mathrm{weak}}_{\mathbb{B}}(A).$$

In particular, the weak $\mathbb{B}$-density property holds.

Lemma

Let $\mathbb{B}$ denote random forcing.

1. *There is a Borel subset A of ${}^{\omega}2$ and an $x \in A$ such that $\liminf_n \mu_n(x, A) \in (0, 1)$ and x is a weak $\mathbb{B}$-density point of A.*
2. *There is a Borel A of ${}^{\omega}2$ and an $x \in A$ such that $\liminf_n \mu_n(x, A) \in (0, 1)$ and x is not a weak $\mathbb{B}$-density point of A.*

Definition (Elżbieta Wagner (1981))

Let $\mathcal{I}$ be an ideal on a space X. The sequence $\langle f_n | n < \omega \rangle$ of functions with domain X converges with respect to I to the function f with domain X if for every sequence $\langle n_\ell | \ell < \omega \rangle$ there is a sequence $\langle \ell_i | i < \omega \rangle$ such that the sequence $\langle f_{\ell_i} | i < \omega \rangle$ converges to I-almost everywhere to f on X.

Observation ([009Wo])

If $\mathcal{I}$ is either the meagre sets or the null sets, TFAE:

1. *x is an $\mathcal{I}$-density point of an $\mathcal{I}$-regular set A,*
2. *For any decreasing to zero sequence of real numbers $\{t_n\}_{n\in\omega}$, there is $\{n_m\}_{m\in\omega}$ such that the sequence $\left\{\chi_{(A-x)/t_{n_m}\cap[1,1]}\right\}_{m\in\omega}$ of characteristic functions converges $\mathcal{I}$-almost everywhere on $[1,1]$ to $\chi_{[1,1]}$,*
3. *Given $\{t_n\}_{n\in\omega}$, a decreasing to zero sequence of real numbers fulfilling condition $\sup_{n\to\infty} t_n/t_{n+1} < \infty$, for every $\{n_m\}_{m\in\omega}$ there is $\{m_p\}_{p\in\omega}$ such that*

$$\left\{\chi_{(A-x)/t_{n_{m_p}}\cap[-1,1]}\right\}_{p\in\omega}$$

converges to $\chi_{[1,1]}$ $\mathcal{I}$-almost everywhere on $[1,1]$.

Definition (Wiesława Poreda, Elżbieta Wagner-Bojakowska, Władysław Wilczyński (1981))

Let $\mathcal{I}$ be an ideal on $\mathbb{R}$ and $A \subset \mathbb{R}$. A real x is an $\mathcal{I}$-density point of A if $\langle \chi_{n(A-x)\cap[-1,+1]} | n < \omega \rangle$ converges with respect to $\mathcal{I}$ to the function which has the constant value 1.
A dispersion point of A is a density point of $\mathbb{R} \setminus A$.

Observation

For all $A \subset \mathbb{R}$ no real is both a density point and a dispersion point of A.

Observation

The density theorem holds for the ideal of meagre sets.

Proposition (Müller, Schlicht, Schrittesser, W. (2018))

There is a $T \in \mathbb{V}$ *such that every* $x \in {}^{\omega}2$ *is a (countable)-dispersion point of* $[T]$*.*

Corollary

The analogue of the Lebesgue Density Theorem fails for many forcings, for example Silver-forcing, Sacks-forcing and anything in between, for example E_0*-forcing or willow-tree forcing.*

Consider the following tree:

$$T := \{t \in {}^{\omega}2 | \forall i < \mathrm{dom}(t)(t(i) = 0 \vee \exists n < i : 2^n = i)\}.$$

Now suppose towards a contradiction that there is an $x \in {}^{\omega}2$ which is not a dispersion point for the ideal of countable sets. This means that that there is an increasing sequence $\langle n_i | i < \omega \rangle$ of natural numbers such that for all increasing sequences $\langle k_i | i < \omega \rangle$ the set

$$B := \limsup_{i<\omega} [T/(x \restriction n_{k_i})] \text{ is uncountable.}$$

Let $\langle n_i | i < \omega \rangle$ be a sequence as above. In particular this statement holds simply for the sequence given by $k_i := i$ for all natural numbers i. Let $C_{\{i,j\}} := [T/(x \restriction n_i)] \cap [T/(x \restriction n_j)]$ for $\{i, j\} \in [\omega]^2$.

We have

$$B \subset \bigcup_{\{i,j\} \in [\omega]^2} C_{\{i,j\}}.$$

By the pigeonhole principle there has to be $\{i,j\} \in [\omega]^2$ such that $C_{\{i,j\}}$ is uncountable. Suppose without loss of generality that $i < j$. In particular $C_{\{i,j\}}$ has to contain at least three elements, call them a, b, c. Recall that for finite sequences x and y we let $x \wedge y$ denote the longest common initial segment. Then $\{a \wedge b, a \wedge c, b \wedge c\}$ is a pair $\{s,t\}$ such that, without loss of generality, $s \subsetneq t$. Both s and t are splitting nodes in both $T/(x \restriction n_i)$ and $T/(x \restriction n_j)$. Let $k_0 := \mathrm{lh}(s)$ and $k_1 := \mathrm{lh}(t)$.

There are natural numbers k_i for $i \in 6 \setminus 2$ such that

$$n_i + k_0 = 2^{k_2},$$
$$n_j + k_0 = 2^{k_3},$$
$$n_i + k_1 = 2^{k_4},$$
$$n_j + k_1 = 2^{k_5}.$$

Therefore $n_j - n_i = 2^{k_3} - 2^{k_2} = 2^{k_5} - 2^{k_4}$, i.e. $2^{k_3} + 2^{k_4} = 2^{k_2} + 2^{k_5}$. As every natural number has a unique represenation in the binary system this implies $(k_2 = k_3 \wedge k_4 = k_5) \vee (k_2 = k_4 \wedge k_3 = k_5)$. As $i < j$ we have $n_i < n_j$ and therefore $k_2 \neq k_3$. Therefore $(k_2 = k_4 \wedge k_3 = k_5)$ and hence $k_0 = k_1$, a contradiction.

Question

Are there weak density points whose density exists, yet is less than 1?

Question

Is there a Borel set $A \subset {}^{\omega}2$ *such that* $D^{\mathbb{P}}_{\mathrm{tr}}(A)$ *fails to be disjoint from* $D^{\mathbb{P}}_{\mathrm{tr}}({}^{\omega}2 \setminus A)$

Question

With $A_x := \{y \in \mathbb{R} \mid \langle x, y\rangle \in A\}$, *does the Lebesgue density theorem hold for any of the ideals*

$$\{A \subset \mathbb{R}^2 \mid \{x \in \mathbb{R} \mid A_x \text{ is not meagre.}\} \text{ is null.}\},$$
$$\{A \subset \mathbb{R}^2 \mid \{x \in \mathbb{R} \mid A_x \text{ is not null.}\} \text{ is meagre.}\}?$$

Question

How can one generalise the Lebesgue density theorem to non-separable spaces?

Thank you for your attention!

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