Block Sequences with Projections into a Sequence of Happy Families

Heike Mildenberger

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The k-valued blocks Fin_k

Definition

Let $k \in \omega \setminus \{0\}$ unless stated otherwise.

(1) For $p \colon \omega \to k+1$ we let $\mathrm{supp}(p) = \{n \in \omega : p(n) \neq 0\}$.

$$\mathrm{Fin}_k = \{p \colon \omega \to k+1 : \mathrm{supp}(p) \text{ finite } \wedge k \in \mathrm{range}(p)\}.$$

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(2) $\mathrm{Fin}_{[1,k]} = \bigcup_{j=1}^{k} \mathrm{Fin}_j$.

(3) For $a, b \in \mathrm{Fin}_k$, we let $a < b$ denote $\mathrm{supp}(a) < \mathrm{supp}(b)$, i.e., $(\forall m \in \mathrm{supp}(a))(\forall n \in \mathrm{supp}(b))(m < n)$. A finite or infinite sequence $\langle a_i : i < m \leq \omega\rangle$ of elements of Fin_k is in block-position if for any $i < j < m$, $a_i < a_j$. The set $(\mathrm{Fin}_k)^\omega$ is the set of ω-sequences in block-position, also called block sequences. For $n \geq 1$, the set $[\mathrm{Fin}_k]^n_<$ is the set of n-sequences in block-position over Fin_k.

Two operations on Fin_j

Definition

(4) For $k \geq 1$, $a, b \in \mathrm{Fin}_k$, we define the partial semigroup operation $+$ as follows: If $\mathrm{supp}(a) < \mathrm{supp}(b)$, then $a + b \in \mathrm{Fin}_k$ is defined. We let $(a + b)(n) = a(n) + b(n)$. Otherwise $a + b$ is undefined. Thus $a+b = a \restriction \mathrm{supp}(a) \cup b \restriction \mathrm{supp}(b) \cup 0 \restriction (\omega \setminus (\mathrm{supp}(a) \cup \mathrm{supp}(b)))$.

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(5) For any $k \geq 2$ we define on Fin_k the Tetris operation: $T \colon \mathrm{Fin}_k \to \mathrm{Fin}_{k-1}$ by $T(p)(n) = \max\{p(n) - 1, 0\}$.

Definition

(6) Let $B \subseteq \mathrm{Fin}_k$ be min-unbounded. We let

$$\begin{aligned}\mathrm{TFU}_k(B) = &\{T^{(j_0)}(b_{n_0}) + \cdots + T^{(j_\ell)}(b_{n_\ell}) \ : \\ &\ell \in \omega \setminus \{0\}, b_{n_i} \in B, b_{n_0} < \cdots < b_{n_\ell}, \\ &j_i \in k, \exists r \leq \ell j_r = 0\}\end{aligned}$$

be the partial subsemigroup of Fin_k generated by B. We call B a TFU_k-set if $B = \mathrm{TFU}_k(B)$.

The condensation order

Definition

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(8) We define the past-operation: Let $\bar{a} \in (\mathrm{Fin}_k)^\omega$ and $p \in \mathrm{Fin}_k$.

$$(\bar{a} \text{ past } p) = \langle a_i \ : \ i \geq i_0 \rangle$$

with $i_0 = \min\{i \ : \ \mathrm{supp}(a_i) > p\}$.

Lemma

If there is $\bar{c} \sqsubseteq_k \bar{a}, \bar{b}$, then there is a largest one and it can be computed by finite initial segments.

Proof.

We define a well-order (of type ω) $\leq_{\mathrm{lex,Fin_k}}$ on the set Fin_k via $a <_{\mathrm{lex,Fin_k}} b$ if $\max(\mathrm{supp}(a)) < \max(\mathrm{supp}(b))$ or $(\max(\mathrm{supp}(a)) = \max(\mathrm{supp}(b))$ and there is an m such that $a \restriction m = b \restriction m$ and $a(m) > b(m)$. For a non-empty set $X \subseteq \mathrm{Fin}_k$ we let $\min_{\mathrm{Fin}_k}(X)$ be the $\leq_{\mathrm{lex,Fin}_k}$-least element of X. We let

$$\begin{aligned} c_0 &= \min_{\mathrm{lex,Fin}_k} (\mathrm{TFU}_k(\bar{a}) \cap \mathrm{TFU}_k(\bar{b})), \\ c_{n+1} &= \min_{\mathrm{lex,Fin}_k} (\mathrm{TFU}_k(\bar{a} \text{ past } c_n) \cap \mathrm{TFU}_k(\bar{b} \text{ past } c_n)) \end{aligned}$$

A subspace of $(\mathrm{Fin}_k)^\omega$–Fixing PP

Definition

We fix parameters as follows. Let $k \geq 1$. Fix $P_{\min}, P_{\max} \subseteq \{1, \ldots, k\}$. Let $PP = \{(i, x) : x \in \{\min, \max\}, i \in P_x\}$ and let

$$\bar{\mathcal{R}} = \{(\iota, \mathcal{R}_\iota) : \iota \in PP\}$$

be a PP-sequence of pairwise nnc Ramsey ultrafilters (pairwise nnc selective coideals, i.e. happy families, would suffice for the pure decision property and properness). We also name the end segments for $1 \leq j \leq k$:

$$\bar{\mathcal{R}} \restriction \{j, \ldots, k\} = \{(\iota, \mathcal{R}_\iota) : \iota = (i, x) \in PP \wedge i \in \{j, \ldots, k\}\}.$$

A subspace of $(\mathrm{Fin}_k)^\omega$

Definition

We let $(\mathrm{Fin}_k)^\omega(\bar{\mathcal{R}})$ denote the set of Fin_k-blocksequences $\bar{a}$ with the following properties:

$$
\begin{aligned}
&(\forall i \in P_{\min})\{\min(a_n^{-1}[\{i\}]) : n \in \omega\} \in \mathcal{R}_{i,\min} \wedge \\
&(\forall i \in P_{\max})\{\max(a_n^{-1}[\{i\}]) : n \in \omega\} \in \mathcal{R}_{i,\max} \wedge \\
&(\forall s \in \mathrm{TFU}_k(\bar{a}))\big(\min(s^{-1}[\{1\}]) < \min(s^{-1}[\{2\}]) < \cdots < \min(s^{-1}[\{k\}]) \\
&\qquad \max(s^{-1}[\{k\}]) < \max(s^{-1}[\{k-1\}]) < \cdots < \max(s^{-1}[\{1\}])\big).
\end{aligned} \tag{0.1}
$$

If $(i, x) \in \{1, \ldots, k\} \times \{\min, \max\} \setminus PP$, we leave the term $x(s^{-1}[\{i\}])$ out of the equation (0.1).

We do not localise to a filter

Lemma

There are $\sqsubseteq_k^*$-incompatible elements in $(\mathrm{Fin}_k)^\omega(\bar{\mathcal{R}})$. Indeed, there are $\bar{a}$, $\bar{b} \in (\mathrm{Fin}_k)^\omega(\bar{\mathcal{R}})$ such that for any $j = 0, \dots, k-1$ the Fin_{k-j}-block-sequences $T^{(j)}[\bar{a}]$ and $T^{(j)}[\bar{b}]$ are $\sqsubseteq_{k-j}^*$-incompatible.

A common strengthening of a theorem by Gowers and a theorem by Blass

The special case of $PP = \{(1, \min), (1, \max)\}$ was proved by Blass in 1987, the case $PP = \emptyset$ and arbitrary finite k by Gowers in 1992.

Theorem

Let k, PP, $\bar{\mathcal{R}}$ be as above. Let $\bar{a} \in (\mathrm{Fin}_k)^\omega(\bar{\mathcal{R}})$ and let c be a colouring of $\mathrm{TFU}_k(\bar{a})$ into finitely many colours. Then there is a $\bar{b} \sqsubseteq_k \bar{a}$, $\bar{b} \in (\mathrm{Fin}_k)^\omega(\bar{\mathcal{R}})$ such that $\mathrm{TFU}_k(\bar{b})$ is c-monochromatic.

Sketch: Proof via Galvin-Glazer technique

Definition

Given k, $P_{\min}$, $P_{\max}$ and $\bar{\mathcal{R}}$ as above, we define

$$\gamma(\mathrm{Fin}_k(\bar{\mathcal{R}})) = \{\mathcal{U} \,:\, \mathcal{U} \text{ is a min-unbounded ultrafilter over } \mathrm{Fin}_k \\ (\forall i \in P_{\min})(\hat{\min}_i(\mathcal{U}) = \mathcal{R}_{i,\min}) \wedge \\ (\forall i \in P_{\max})(\hat{\min}_i(\mathcal{U}) = \mathcal{R}_{i,\max})\},$$

endowed with the topology given by the basic open sets

$$\Big\{\{\mathcal{U} \in \gamma(\mathrm{Fin}_k(\bar{\mathcal{R}})) \,:\, A \in \mathcal{U}\} \,:\, A \subset \mathrm{Fin}_k, \\ \{x(s^{-1}[\{i\}]) \,:\, s \in A\} \in \mathcal{R}_{i,x}\Big\}.$$

The space $\gamma(\mathrm{Fin}_k(\bar{\mathcal{R}}))$ a compact Hausdorff space.

Strict PP's

For work with semigroups of ultrafilters we temporarily have to choose PP in a narrower sense. The reason is the claim part of Def. and Lemma below. We do not know how to handle missing i in the sequence of $\mathcal{R}_{i,\min}$'s or in the sequence of $\mathcal{R}_{i,\max}$'s in the claim.

Definition

For any $k \geq 1$, a reservoir of indices PP of the strict form is one of the following three types: $PP = \{(i, \min), (i, \max) \, : \, 1 \leq i \leq k\}$, $PP = \{(i, \min) \, : \, 1 \leq i \leq k\}$, $PP = \{(i, \min) \, : \, 1 \leq i \leq k\}$.

Definition and Lemma

Again we work with strict PP. For $2 \leq j \leq k$, we write
$T[X] = \{T(a) \, : \, a \in X\}$ *for* $X \subseteq \mathrm{Fin}_j(\bar{\mathcal{R}} \restriction \{k-j+1, k\})$ *and*
$T[\bar{a}] = \langle T(a_n) \, : \, n \in \omega \rangle$ *for* $\bar{a} \in (\mathrm{Fin}_j)^\omega(\bar{\mathcal{R}} \restriction \{k-j+1, k\})$.
The lift of the tetris operation

$$\dot{T} \colon \gamma(\mathrm{Fin}_j(\bar{\mathcal{R}} \restriction \{k-j+1, \dots, k\})) \to \gamma(\mathrm{Fin}_{j-1}(\bar{\mathcal{R}} \restriction \{k-j+2, \dots, k\}))$$

is defined via

$$\dot{T}(\mathcal{U}) = \{T[X] \, : \, X \in \mathcal{U}\}.$$

A lift of the partial semigroup operation $+$

Definition and Lemma

Let k, PP and $\bar{\mathcal{R}}$ be as above, with strict PP. We define $\dot{+}$ on $(\bigcup_{j=1}^{k} \gamma(\mathrm{Fin}_j)(\bar{\mathcal{R}} \restriction \{k-j+1,\ldots,k\}))^2$ as follows.

$$\dot{+} \colon \gamma(\mathrm{Fin}_i(\bar{\mathcal{R}} \restriction \{k-i+1,\ldots,k\})) \times \gamma(\mathrm{Fin}_j(\bar{\mathcal{R}} \restriction \{k-j+1,\ldots,k\}))$$
$$\to \gamma\mathrm{Fin}_{\max\{i,j\}}(\bar{\mathcal{R}} \restriction \{k-\max(i,j)+1,\ldots,k\})$$

is defined as

$$\mathcal{U} \dot{+} \mathscr{V} = \Big\{ X \subseteq \mathrm{Fin}_{\max\{i,j\}}(\bar{\mathcal{R}} \restriction \{k-\max(i,j)+1,\ldots,k\})$$
$$: \{s : \{t : s+t \in X\} \in \mathscr{V}\} \in \mathcal{U} \Big\}.$$

Diagonal lower bounds

Lemma

let k, PP, $\bar{\mathcal{R}}$ be as above, not necessarily strict. Here the strict form of PP is not needed. Any $\sqsubseteq_k$-descending sequence $\langle \bar{c}_n : n \in \omega \rangle$ in $(\mathrm{Fin}_k)^\omega(\bar{\mathcal{R}})$ has a diagonal lower bound $\bar{b} \in (\mathrm{Fin}_k)^\omega(\bar{\mathcal{R}})$

$$(\forall n \in \omega)((\bar{b} \text{ past } b_n) \sqsubseteq_k \bar{c}_{\max(\mathrm{supp}(b_n))+1}).$$

such that each b_{n+1} is an element of $\{c_{\ell_{n+1},m} : m \in \omega\}$ for some $\ell_{n+1} > \max(\mathrm{supp}(b_n))$ and b_0 is an element of $\{c_{\ell_0,m} : m \in \omega\}$ for some ℓ_0.

A k-sequence of very good idempotent ultrafilters

Lemma

(Lemma 2.24, Todorcevic, Ramsey Spaces) Let k, PP, $\bar{\mathcal{R}}$ be as above, with full PP. For any $k \geq j \geq 1$, and $\bar{a} \in (\mathrm{Fin}_k)^\omega(\bar{\mathcal{R}})$ there is an idempotent $\mathcal{U}_j \in \gamma(\mathrm{Fin}_j(\bar{\mathcal{R}} \restriction \{k+j-1, \dots, k\}))$ such that for all $1 \leq i \leq j \leq k$

(1) $\mathcal{U}_j \dot{+} \mathcal{U}_i = \mathcal{U}_j$,

(2) $\dot{T}^{(j-i)}(\mathcal{U}_j) = \mathcal{U}_i$.

(3) $T^{(i-1)}(\bar{a}) \in \mathcal{U}_{k-i+1}$.

A useful notion of forcing

Definition

We let k, PP, $\bar{\mathcal{R}}$ be as above, not necessarily strict. In the Gowers–Matet forcing with $\bar{\mathcal{R}}$, $\mathbb{M}_k(\bar{\mathcal{R}})$, the conditions are pairs $(s, \bar{c})$ such that $s \in \mathrm{Fin}_k$ and $\bar{c} \in (\mathrm{Fin}_k)^\omega(\bar{\mathcal{R}})$ and $\mathrm{supp}(s) < \mathrm{supp}(c_0)$.
The forcing order is: $(t, \bar{b}) \leq (s, \bar{a})$ if $t = s + s'$ and $s' \in \mathrm{TFU}_k(\bar{a})$ and $\bar{b} \sqsubseteq_k (\bar{a} \text{ past } s')$

Definition

For $(s, \bar{a}), (t, \bar{b}) \in \mathbb{M}_k(\bar{\mathcal{R}})$ and $n \in \omega$ we let $(s, \bar{a}) \leq_n (t, \bar{b})$ if $s = t$ and $a_i = b_i$ for $i < n$.

Lemma

$\mathbb{M}_k(\bar{\mathcal{R}})$ *has the pure decision property, i.e., for any* $\varphi \in \mathcal{L}(\in)$, $(s, \bar{a}) \in \mathbb{M}_k(\bar{\mathcal{R}})$ $\exists (s, \bar{b}) \leq (s, \bar{a})$ $((s, \bar{b}) \Vdash \varphi \vee (s, \bar{b}) \Vdash \neg\varphi)$.

Stepping up to finite dimensions

Since the space $(\mathrm{Fin}_k)^\omega(\bar{\mathcal{R}})$ is stable, we can step up the Milliken–Taylor style to higher finite arities:

Theorem

Let $n \in \omega \setminus \{0\}$ and $\bar{a} \in (\mathrm{Fin}_k)^\omega(\bar{\mathcal{R}})$ and let c be a colouring of $[\mathrm{TFU}_k(\bar{a})]^n_<$ into finitely many colours. Then there is a $\bar{b} \sqsubseteq_k \bar{a}$, $\bar{b} \in (\mathrm{Fin}_k)^\omega(\bar{\mathcal{R}})$ such that $[\mathrm{TFU}_k(\bar{b})]^n_<$ is c-monochromatic.