A Parallel Metrization Theorem

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In this talk I shall present a solution of one question asked on Mathoverflow by `user116515`.

The question concerns parallel sets in metric spaces.

Definition

Two non-empty sets A, B in a metric space (X, d) are called *parallel* if

$$d(a, B) = d(A, B) = d(A, b) \quad \text{for any } a \in A \text{ and } b \in B.$$

Here $d(A, B) = \inf\{d(a, b) : a \in A,\ b \in B\}$
and $d(x, B) = d(B, x) := d(\{x\}, B)$ for $x \in X$.

Observe that two closed parallel sets A, B is a metric space are either disjoint or coincide.

A MO problem on parallel metrics

Definition

Let $\mathcal{C}$ be a family of closed subsets of a topological space X. A metric d on X is called *$\mathcal{C}$-parallel* if any two sets $A, B \in \mathcal{C}$ are parallel with respect to the metric d.

A family $\mathcal{C}$ of subsets of X is called a *compact cover* of X if $X = \bigcup \mathcal{C}$ and each set $C \in \mathcal{C}$ is compact.

Problem (MO)

For which compact covers $\mathcal{C}$ of a topological space X the topology of X is generated by a $\mathcal{C}$-parallel metric?

Example

The Euclidean metric on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| \le 1\}$ is parallel with respect to the cover $\mathcal{C} = \{C_r : r \in [0,1]\}$ of $\mathbb{D}$ by circles $C_r = \{z \in \mathbb{C} : |z| = r\}$.

Continuity of families

A metric generating the topology of a given topological space is called *admissible*.

Let $\mathcal{C}$ be a cover $\mathcal{C}$ of a set X. A subset $A \subset X$ is called *$\mathcal{C}$-saturated* if A coincides with its *$\mathcal{C}$-saturation*

$$[A]_{\mathcal{C}} := \bigcup\{C \in \mathcal{C} : A \cap C \neq \emptyset\}.$$

A family $\mathcal{C}$ of subsets of a topological space X is called

- *lower semicontinuous* if for any open set $U \subset X$ its $\mathcal{C}$-saturation $[U]_{\mathcal{C}}$ is open in X;
- *upper semicontinuous* if for any closed set $F \subset X$ its $\mathcal{C}$-saturation $[F]_{\mathcal{C}}$ is closed in X;
- *continuous* if $\mathcal{C}$ is both lower and upper semicontinuous;
- *disjoint* if any distinct sets $A, B \in \mathcal{C}$ are disjoint.

A parallel metrization theorem

Main Theorem

For a compact cover $\mathcal{C}$ of a metrizable topological space X the following conditions are equivalent:

1. *the topology of X is generated by a $\mathcal{C}$-parallel metric;*
2. *the family $\mathcal{C}$ is disjoint and continuous.*

https://mathoverflow.net/questions/284544/making-compact-subsets-parallel

Thank You!

Děkuji!