Nonmeasurable images in Polish space with respect to σ-ideals with Borel base

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Notation and Terminology

Let X is a Polish space and $I \subseteq \mathscr{P}(X)$ s.t

- I is σ-ideal with a Borel base and
- I contains all singletons,

then (X, I) is Polish ideal space
Let $\mathcal{B}_+(I) = Borel(X) \setminus I$ be set of all I-positive Borel sets.
$Perf(X)$ stands for set of all perfect subsets of X

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Definition (Cardinal coefficients)

Let X - Polish space and $I \subseteq \mathscr{P}(X)$ be σ-ideal and $\mathscr{F} \subset I$ let

$$cov(\mathcal{F}) = min\{|\mathscr{A}| : \mathscr{A} \subset \mathcal{F} \wedge \bigcup \mathscr{A} = X\}$$

$$cov_h(\mathcal{F}) = min\{|\mathscr{A}| : \mathscr{A} \subset \mathcal{F} \wedge (\exists B \in \mathcal{B}_+(I)) \bigcup \mathscr{A} = B\}$$

$$cof(I) = min\{|\mathscr{B}| : \mathscr{B} \subseteq I \wedge (\forall A \in I)(\exists B \in \mathscr{B})\ A \subseteq B\}$$

$$Cof(I) = \min\{|\mathscr{B}| : \mathscr{B} \subseteq \mathcal{B}_+(I) \wedge (\forall A \in \mathcal{B}_+(I))(\exists B \in \mathscr{B})(B \subseteq A)\}$$

$\mathscr{N}$ σ-ideal of null sets and $\mathscr{M}$ σ-ideal of all meager subsets of X.
$cov(\mathscr{M}) = cov_h(\mathscr{M})$, $cov(\mathscr{N}) = cov_h(\mathscr{N})$.

Theorem (Cichoń-Kamburelis-Pawlikowski)

If I is c.c.c. σ-ideal with Borel base then $cof(I) = Cof(I)$

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Complete I-nonmeasurability

Definition

Let (X, I) be Polish ideal space. We say that $A \subseteq X$ is completely I-nonmeasurable in X iff

$$(\forall B \in \mathcal{B}_+(I))\ A \cap B \neq \emptyset \wedge A^c \cap B \neq \emptyset.$$

- $A \subseteq X$ is complete $[X]^{\leq\omega}$-nonmeasurable iff A is Bernstein subset of X,
- $A \subseteq [0,1]$ is complete $\mathscr{N}$-nonmeasurable iff $\lambda_*(A) = 0$ and $\lambda^*(B) = 1$,
- $A \subseteq X$ is complete $\mathscr{M}$-nonmeasurable if $\emptyset \neq U \subseteq X$ then $A \cap U$ does not have Baire property.

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Theorem

Let $X \subseteq X_0$ and $\{Y_\alpha : \alpha < \mathfrak{c}\}$ be a Polish subspaces
Assume that $\mathfrak{c}$ is regular cardinal number.
If $\{f_\alpha : \alpha < \mathfrak{c}\}$ be a family of functions such that for any $\alpha < \mathfrak{c}$

1. $f_\alpha[X] = Y_\alpha$,
2. *for any* $y \in \bigcup_{\alpha<\mathfrak{c}} Y_\alpha$ *we have* $f_\alpha^{-1}[y] \in [X]^{<\mathfrak{c}}$.

Then there exists a subset $A \subseteq X$ such that for any $\alpha < \mathfrak{c}$ $f_\alpha[A]$ is a Bernstein set in Y_α.

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Corollary

There is subset $A \subset S^1$ of the unit circle that for any projection π on real line $l \subseteq \mathbb{R}^2$ on the real plane of the set A is a Bernstein set in $\pi[S^1]$.

Thus we have negative answer for

> *[asked Aug 3 '11 at 7:51 simon 162]* Suppose A is contained in the unit square of R^2, and the projection of A on any line outside the unit square is not Lebesgue measurable in R. Does that imply that A is not Lebesgue measurable in the plane?

Moreover, our answer is valid for measure and category simultaneously.

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Example

Let

- $\mathcal{F} \subseteq P(\omega)$ - Frechet filter,
- $X = \omega^\omega$, $Y_C = \omega^C$ where $C \in \mathcal{F}$,
- $\omega^\omega \ni x \mapsto f_C(x) = x \restriction C \in \omega^C$.

Then by the Theorem there is $A \subset \omega^\omega$ such that each image $f_c[A]$ is a Bernstein subset of ω^C.

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Remark

If we consider any function $f : X \to X_0$ such that $f[X]$ is a Polish space, $A \subseteq X$ Bernstein set then

1. *if preimage of any singleton of $f[X]$ contains a perfect set then $f[A] = f[X]$,*
2. *if f is continuous then $f[A]$ contains some Bernstein set in $f[X]$ (because any preimage of perfect set in $f[X]$ contains perfect set in X).*

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Theorem

Let (X_0, I) be Polish ideal space and let $X \subseteq X_0$ be I-positive Borel subset. *Let $\mathcal{F}$ be a family with the following properties:*

1. $(\forall f \in \mathcal{F})(f[X] \subseteq X_0$ *is Polish space*$)$,
2. $(\forall f \in \mathcal{F})(f : X \to X_0 \wedge I_f \subseteq P(f[X])$ *be σ-ideal with Borel base on* $f[X])$,
3. $|\mathcal{F}| \leq \sup\{Cof(I_f) : f \in \mathcal{F}\}$,
4. $\sup\{Cof(I_f) : f \in \mathcal{F}\} \leq \min\{|Z| : Z \subseteq X_0 \wedge (\exists f \in \mathcal{F})(\exists B \in Bor(f[X]) \setminus I_f)(\exists \mathcal{F}_0 \subseteq \mathcal{F})(|\mathcal{F}_0| \leq |Z| \wedge f^{-1}[B] \subseteq \bigcup\{h^{-1}[Z] : h \in \mathcal{F}_0\})\}$.

Then there exists subset A of X such that for any $f \in \mathcal{F}$ the image $f[A]$ is completely I_f-nonmeasrable in $f[X]$.

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1. *for every* $f \in \mathcal{F}$ *the image* $f[X]$ *is Borel subset of* X_0 *and* $I_f \subseteq P([f[X]])$ *is* σ*-ideal with Borel base in* $f[X]$*,*
2. $|\mathcal{F}| \leq \max\{Cof(I), \sup\{Cof(I_f) : f \in \mathcal{F}\}\}$,
3. *there is set* $Z \in I$ *such that* $Cof(I) \leq cov(\{f^{-1}[\{d\}] : f \in \mathcal{F} \wedge d \in X_0 \setminus Z\}, I)$,
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Corollary

Assume MA. If $I \in \{\mathscr{N}, \mathscr{M}\}$ *is a* σ*-ideal defined on Cantor space and* $X \subset 2^\omega$ *be a Borel* I*-positive. If* $\mathcal{F}$ *with at most size equal to* $\mathfrak{c}$ *and for any* $f \in \mathcal{F}$ $rng(f)$ *is Borel and* $I_f \in \{\mathscr{N}, \mathscr{M}\}$ *then the above two Theorems are true.*

In the Mathoverflow webpage [2] the user Gowers gives positive answer for the following question

> *[Gerald Edgar Aug 3 '11 at 13:57]* (a) All projections but two are non-measurable? Or: (b) Projections in uncountably many directions measurable and projections in uncountably many other directions non-measurable?

The user of Mathoverflow asked:

> *[answered Aug 3 '11 at 14:47 gowers]* I don't know what happens if we ask for continuum many measurable projections and continuum many non-measurable projections ...

Theorem

Let $\mathfrak{c}$ be regular, X, Y be Polish spaces and

- $\{Y_\alpha : \alpha \in Y\}$ *be a familly of Polish spaces,*
- $\{f_\alpha : \alpha \in Y\}$ *be a family functions such that for all distinct* $\alpha, \beta \in Y$
  - $\forall y \in Y_\alpha \; |f_\alpha^{-1}[y]| = \mathfrak{c}$
  - $\forall y \in Y_\alpha$ *and* $y' \in Y_\beta \; |f_\alpha[y] \cap f_\beta[y']| < \mathfrak{c}$.

Then there exists a subset $A \subseteq X$ and disjoint Bernstein sets $F, G \subseteq Y$ such that $Y = F \cup G$ and

$$F = \{\alpha \in Y : f_\alpha[A] = Y_A\}$$

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Fact

Let $n \geq 2$ *be a fixed integer then every projection* π *of the Lusin set* $A \subseteq B(0,1) \subset \mathbb{R}^n$ *into tangent hyperplane* l *to* $B(0,1)$ *is Lusin set in* $\pi[B(0,1)]$*. The same result is true if we replace Lusin set by Sierpiński set.*

Fact

It is relatively consistent with ZFC theory that $\neg CH$ *and for every integer* $n \geq 2$ *there exists Baire nonmeasurable subset* A *of the cardinality less than* $\mathfrak{c}$ *of the unit ball* $B \subseteq \mathbb{R}^n$ *such that projection* $\pi[A]$ *into any tangent to* B *hyperplane has not Baire property. The same result is true in the case of Lebesgue measure.*

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Theorem

Let X be a compact Polish space and $G \subseteq \mathscr{H}(X)$ be uncountable G_δ subset of $\mathscr{H}(X)$. Let $B \subseteq X$ be a comeager subset of X. Then there are perfect subsets $P \subseteq X$ and $Q \subseteq G$ such that for every homeomorphism $f \in Q$ of X we have $P \subseteq f[B]$.

Theorem

Let $D \subseteq \mathbb{R}^2$ be a unit disc with center in origin coordinates and $B \subseteq D$ a comeager (or $D \setminus B$ is null) set in D. Then there are perfect set of directions R on $bd(D)$ and $P, Q \subseteq [-1, 1]$ such that

$$(\forall \alpha \in R)\ (r_\alpha[P \times Q] \subseteq B),$$

where r_α is rotation by α over origin of the real plane $\mathbb{R}^2$.

Theorem

Let $n \geq 2$ and $B_n \subseteq \mathbb{R}^n$ be a n-dimensional unit ball. Let us assume that $E \subseteq B$ a comeager (or $B_n \setminus E$ is null) set in B_n. Then there are perfect set R in $D = bd(B_n)$, non-meager (non-null) $P \subseteq B_{n-1}$ and $Q \subseteq [-1, 1]$ such that

$$(\forall \alpha \in R)\ (r_\alpha[P \times Q] \subseteq B_n),$$

where r_α is rotation of α to the vector $(1, 0, \ldots, 0) \in \mathbb{R}^n$ over origin of the euclidean space $\mathbb{R}^n$.

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Thank You

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Thank You