Luzin and Sierpiński sets meet trees

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A tree T on ω is called

- Sacks tree or perfect tree, denoted by $T \in \mathbb{S}$, if for each node $s \in T$ there is $t \in T$ such that $s \subseteq t$ and $|succ(t)| \geq 2$;
- Miller tree or superperfect tree, denoted by $T \in \mathbb{M}$, if for each node $s \in T$ exists $t \in T$ such that $s \subseteq t$ and $|succ(t)| = \aleph_0$;
- Laver tree, denoted by $T \in \mathbb{L}$, if for each node $t \supseteq stem(T)$ we have $|succ(t)| = \aleph_0$;
- complete Laver tree, denoted by $T \in \mathbb{CL}$, if T is Laver and $stem(T) = \emptyset$;

Let $\mathbb{T}$ be a family of trees. Then we define a *tree ideal* t_0 as follows:

Definition 1. *Let $X \subseteq \omega^\omega$. Then*

$$X \in t_0 \Leftrightarrow (\forall T \in \mathbb{T})(\exists T' \subseteq T, T' \in \mathbb{T})(T' \cap X = \emptyset).$$

For example s_0 is the classic Marczewski ideal.
Let us recall a notion of $\mathcal{I}$-Luzin sets.

Definition 2. *Let X be a Polish space and $\mathcal{I}$ be an ideal. Then we call a set $L \subseteq X$ an $\mathcal{I}$-Luzin set if $|L \cap A| < |L|$ for all $A \in \mathcal{I}$.*

For classic ideals of Lebesgue measure zero sets $\mathcal{N}$ and meager sets $\mathcal{M}$ we will call $\mathcal{M}$-Luzin sets generalized Luzin sets and $\mathcal{N}$-Luzin sets generalized Sierpiński sets.
We will consider $\mathcal{I}$-Luzin sets in a context of algebraic properties and tree ideals. We will work on the real line $\mathbb{R}$ with addition. Since $\mathbb{R}$ is σ-compact, it does not contain even bodies of Miller trees. We will tweak the definition a bit by saying that $A \subseteq \mathbb{R}$ belongs to t_0 if $h^{-1}[A]$ belongs to t_0 in ω^ω, where h is a homeomorphism between ω^ω and a subspace of irrational numbers (see [1] for a similar modification in the case of 2^ω). Using a subtle kind of fusion for Miller and Laver trees we will prove that

Lemma 1. *There exists a dense G_δ set G such that for each Miller (resp. Laver or complete Laver) tree T there exists a Miller (resp. Laver or complete Laver) subtree $T' \subseteq T$ such that $G + [T'] \in \mathcal{N}$.*

We will use this result to obtain the following theorem that extends the result achieved in [3].

Theorem 1. *Let $\mathfrak{c}$ be a regular cardinal and $t_0 \in \{s_0, m_0, l_0, cl_0\}$. Then for every generalized Luzin set L and generalized Sierpiński set S we have $L + S \in t_0$.*

Results are available in [2].

References

[1] Kysiak M., Weiss T., Small subsets of the reals and tree forcing notions, Proceedings of American Mathematical Society, vol. 132, nr 1, pp. 251-259, 2003.

[2] Michalski M., Rałowski R., Żeberski Sz., Nonmeasurable sets and union with respect to tree ideals, arXiv:1712.05212 (2017)

[3] Michalski M., Żeberski Sz., Some properties of I-Luzin, Topology and its Applications, 189 (2015), pp. 122-135,