Some games and their topological consequences II

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ICMC-USP (Partially supported by FAPESP)

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Theorem ([1])

The Rothberger game and the point-open game are dual (i.e. ALICE has a winning strategy in one of the games iff BOB has a winning strategy in the other one)

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Then ALICE plays $\mathcal{C}_1$. Let $x_1 = \sigma(\langle C_1 \rangle)$. As before, BOB can select $C_1 \in \mathcal{C}_1$ such that $x_1 \in C_1$.

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Then Alice plays $\mathcal{C}_1$. Let $x_1 = \sigma(\langle C_1 \rangle)$. As before, Bob can select $C_1 \in \mathcal{C}_1$ such that $x_1 \in C_1$. Then Alice plays $\mathcal{C}_2$ and we fix $x_2 = \sigma(\langle C_1, C_2 \rangle)$ and so on.

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The point is, the sequence $\langle x_0, C_0, x_1, C_1, ... \rangle$ is a play of the point-open game where the strategy σ was used.

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Then Alice plays $\mathcal{C}_1$. Let $x_1 = \sigma(\langle C_1 \rangle)$. As before, Bob can select $C_1 \in \mathcal{C}_1$ such that $x_1 \in C_1$. Then Alice plays $\mathcal{C}_2$ and we fix $x_2 = \sigma(\langle C_1, C_2 \rangle)$ and so on.

The point is, the sequence $\langle x_0, C_0, x_1, C_1, ...\rangle$ is a play of the point-open game where the strategy σ was used. Therefore $\bigcup_{n\in\omega} C_n = X$.

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Let us look at the first inning: Alice knows (from the other game) how to select elements from open coverings. But she needs to begin this game playing a point.

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Proof.

Suppose not. Then for every x there is a C_x such that $x \in C_x$ and C_x is not a possible answer from σ. Note that $\mathcal{C} = \{C_x : x \in X\}$ is an open covering. Since $\sigma(\langle \mathcal{C} \rangle) \in \mathcal{C}$, we got a contradiction. $\square$

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Therefore the sequence $\langle \mathcal{C}_0, C_0, \mathcal{C}_1, C_1, \dots \rangle$ is a play of the Rothberger game where σ was used. Therefore $\bigcup_{n \in \omega} C_n = X$.

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Putting all this together:

$$\mathsf{G}_{\mathrm{k}}(\mathcal{A}, \mathcal{B})$$

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Abusing a little bit from the notation, we can denote the Menger game as

$$\mathsf{G}_{\text{fin}}(\mathcal{O}, \mathcal{O})$$

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Suppose that we are now in a different situation. Now, instead of ALICE playing an open covering per inning, suppose that she just gives away the complete sequence of open coverings. Then BOB just has to select one open set from each. When BOB can make a covering this way, we say that the selection principle holds and denote such a property as

$$\mathsf{S}_1(\mathcal{O}, \mathcal{O})$$

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$$\neg \mathsf{S}_1(\mathcal{A},\mathcal{B}) \Rightarrow \text{ALICE} \uparrow \mathsf{G}_1(\mathcal{A},\mathcal{B}) \Rightarrow \text{BOB} \not\uparrow \mathsf{G}_1(\mathcal{A},\mathcal{B})$$

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In most of cases of the pairs $\mathcal{A}$, $\mathcal{B}$, none of the "$\Leftarrow$" implications are true.

Alice $\uparrow \mathsf{G}_1(\mathcal{A}, \mathcal{B}) \not\Rightarrow \neg\mathsf{S}_1(\mathcal{A}, \mathcal{B})$

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As a warming up, think about the following:

Exercise

If X is first countable at p, then Bob *has a winning strategy for* $\mathsf{G}_1(\Omega_p, \Omega_p)$.

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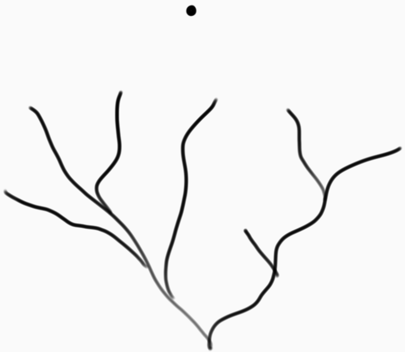
$$\{p\} \cup \omega^{<\omega} \setminus F$$

where F is a finite collection of branches in $\omega^{<\omega}$.

A picture is better than words



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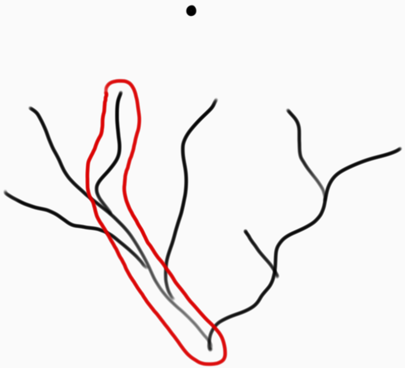

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- and so on.

At the end, the sequence of choices of BOB is a branch in $\omega^{<\omega}$. Thus ALICE wins the game.

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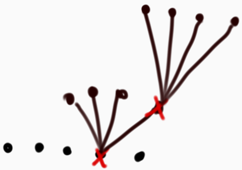

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Let $\langle A_n : n \in \omega \rangle$ be a sequence of elements of Ω_p. By $\mathsf{S}_2(\Omega_p, \Omega_p)$, we can find $\langle \langle a_n, b_n \rangle : n \in \omega \rangle$ where each $\langle a_n, b_n \rangle \subset A_n$ and

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Therefore, selecting $\langle a_n : n \in \omega \rangle$ or $\langle b_n : n \in \omega \rangle$ would be enough. $\square$

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(Let just state everything together: we proved that ALICE has a winning strategy for $\mathsf{G}_1(\Omega_p, \Omega_p)$ and we will prove now that BOB has a winning strategy for the $\mathsf{G}_2(\Omega_p, \Omega_p)$ - this will be enough since this implies $\mathsf{S}_2(\Omega_p, \Omega_p)$ which is equivalent to $\mathsf{S}_1(\Omega_p, \Omega_p)$)

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At the first inning, just pick any two incompatible elements.

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We will make a strategy for Bob that guarantees that at each inning n, the choices of Bob so far contain a set $\{b_1, ..., b_n\}$ such that no branch meets two different elements. Note that this is enough.

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1. there is a b_{n+1} that is not in the same branch as any of the $b_1, ..., b_n$. Bob just picks this point (and anything else);
2. There is no b_{n+1} as above. Note that $A \setminus \{b_1, ..., b_n\} \in \Omega_p$.

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We will make a strategy for $\textsc{Bob}$ that guarantees that at each inning n, the choices of $\textsc{Bob}$ so far contain a set $\{b_1, ..., b_n\}$ such that no branch meets two different elements. Note that this is enough.

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1. there is a b_{n+1} that is not in the same branch as any of the $b_1, ..., b_n$. $\textsc{Bob}$ just picks this point (and anything else);
2. There is no b_{n+1} as above. Note that $A \smallsetminus \{b_1, ..., b_n\} \in \Omega_p$. Therefore there are $c_1, c_2 \in A \smallsetminus \{b_1, ..., b_n\}$ and b_j such that $b_j \subset c_1, c_2$.

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If the space is separable and Bob has a winning strategy for $\mathsf{G}_1(\Omega_p, \Omega_p)$, then p has a local countable base.

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Definition

We say that X is **productively countably tight** at p if for every Y countably tight at q, $X \times Y$ is countably tight at $\langle p, q \rangle$.

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$$\textsc{Bob} \uparrow \mathsf{G}_1(\Omega_p, \Omega_p) \Rightarrow X \text{ is productively countably tight at} p \Rightarrow \mathsf{S}_1(\Omega_p \Omega_p)$$

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Theorem ([3])

If there is a Michael space, then every productively Lindelöf space is Menger.

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