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Reflection cardinals of coloring of graphs

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Winter School in Abstract Analysis 2017
section Set Theory & Topology

(2017 年 02 月 17 日 (07:02 CET) version)

2017 年 2 月 3 日 (于 Mezinárodní Centrum Duchovní Obnovy)

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Reflection cardinal

- $\mathcal{C}$: a class of structures with notions of substructures (notation: $A \leq B$ for "A, $B \in \mathcal{C}$, and A is a substructure of B"), the underlying set (denoted also by A for $A \in \mathcal{C}$) and the cardinality $|\,A\,|$ of the structures $A \in \mathcal{C}$.

- For $A \in \mathcal{C}$, $\mathcal{S}_{<\kappa}(A) = \{B \in \mathcal{C} \,:\, B \leq A, \,|\,B\,| < \kappa\}$.
  Similarly for $\mathcal{S}_{\leq\kappa}(A)$, $\mathcal{S}_{\kappa}(A)$ etc.

- For a property $\mathcal{P}$

$$\mathfrak{Refl}\,(\mathcal{C}, \mathcal{P}) = \min\{\kappa \in \mathrm{Card} \,:\, \text{for any } A \in \mathcal{C} \text{ if } A \models \mathcal{P} \text{ then there are stationarily many } A' \in \mathcal{S}_{<\kappa}(A) \text{ s.t. } A' \models \mathcal{P}\}$$

- We let here $\min\emptyset = \infty$.

Examples (1/3)

- For $\mathcal{C}$ = compact spaces and $\mathcal{P}$: non-metrizable, we can prove in ZFC: $\mathfrak{Refl}(\mathcal{C}, \mathcal{P}) = \aleph_2$ (Alan Dow, 1988).

- $\mathfrak{Refl}(\mathcal{C}, \mathcal{P}) = \aleph_2$ for these $\mathcal{C}$ and $\mathcal{P}$ means:

  (ZFC) *If a compact space X is non-metrizable then X has a non-metrizable subspace of cardinality $\leq \aleph_1$.*

- Dow's theorem is one of the first theorems in topology where the only natural proof is obtained by the method of elementary submodels and the elementary submodel proof was the proof which established the theorem.

Examples (2/3)

Theorem 1 (S.F., H. Sakai, L. Soukup, T. Usuba et al.)

The following are equivalent:

(a) $\mathfrak{Refl}(\mathcal{C},\mathcal{P}) = \aleph_2$ *for* $\mathcal{C}$ = *locally compact spaces and* $\mathcal{P}$: *non-metrizable*

(b) *Fodor-type Reflection Principe (*FRP*)*

- FRP will be defined later.
- FRP implies the total failure of square principle.
- FRP can be forced starting from a model with a strongly compact cardinal.
- Thus $\mathfrak{Refl}(\mathcal{C},\mathcal{P}) = \aleph_2$ for $\mathcal{C}$ and $\mathcal{P}$ as above is consistent (modulo a large cardinal).
- FRP is compatible with any assertions forcable by ccc po (also starting from a model of CH or MM).

Examples (3/3)

- For $\mathcal{C}$ = first countable spaces and $\mathcal{P}$: non-metrizable, the consistency of the equation $\mathfrak{Refl}(\mathcal{C}, \mathcal{P}) = \aleph_2$ is unsolved (Hamburger's problem).

▷ for $\mathcal{C}$ and $\mathcal{P}$ as above, $\mathfrak{Refl}(\mathcal{C}, \mathcal{P}) \leq 2^{\aleph_0}$ is consistent (relative to a large cardinal, A. Dow, F. Tall and W.A.R., Weiss (1990)).

- For $\mathcal{C}$ = topological spaces and $\mathcal{P}$: non-metrizable, $\mathfrak{Refl}(\mathcal{C}, \mathcal{P}) = \infty$ (A. Hajnal and I. Juhász (1976)).

  [For any regular κ, the topological space $\langle \kappa+1, \mathcal{O} \rangle$ with $\mathcal{O} = \mathcal{P}(\kappa) \cup \{\kappa+1 \setminus x \,:\, x \subseteq \lambda \text{ is bounded in } \kappa\}$ witnesses $\mathfrak{Refl}(\mathcal{C}, \mathcal{P}) > \kappa$.]

Reflection cardinals for coloring of graphs

- For a cardinal δ let $\mathfrak{Refl}_{>\delta\text{-}col}$ be the reflection cardinal $\mathfrak{Refl}(\mathcal{C},\mathcal{P})$ for $\mathcal{C}$ = graphs and $\mathcal{P}$: "of coloring number $>\delta$".

▷ $\mathfrak{Refl}_{>\delta\text{-}col} = \min\{\kappa$: for any graph G, if $col(G)>\delta$ then there is $G'\in\mathcal{S}_{<\kappa}(G)$ with $col(G')>\delta\}$

- Let $\mathfrak{Refl}_{>\delta\text{-}chr}$ be the reflection cardinal $\mathfrak{Refl}(\mathcal{C},\mathcal{P})$ for $\mathcal{C}$ = graphs and $\mathcal{P}$: "of chromatic number $>\delta$".

▷ $\mathfrak{Refl}_{>\delta\text{-}chr} = \min\{\kappa$: for any graph G, if $\mathrm{chr}(G)>\delta$ then there is $G'\in\mathcal{S}_{<\kappa}(G)$ with $\mathrm{chr}(G')>\delta\}$

Reflection cardinals for coloring of graphs (2/2)

Lemma 2

For any graph G, we have $\mathrm{chr}(G) \leq col(G)$. There are graphs G with $\mathrm{chr}(G) < col(G)$.

Theorem 3 (S.F., H. Sakai, L. Soukup, T. Usuba et al.)

$\mathfrak{Refl}_{>\omega\text{-}col} = \aleph_2$ *is also equivalent to* FRP*. In particular this equiation is consistent (modulo a large cardinal).*

Theorem 4 (P. Erdős and A. Hajnal 1966)

$\mathfrak{Refl}_{>\omega\text{-}chr} = \aleph_2$ *is inconsistent!*
In ZFC*, it is provable that* $\mathfrak{Refl}_{>\omega\text{-}chr} > \beth_\omega$.

Problem 1 Does $\delta > \omega$ analog of Erdős-Hajnal Theorem hold?

Main objective of the talk

- It is not obvious in which relation $\mathfrak{Refl}_{>\delta\text{-}col}$ and $\mathfrak{Refl}_{>\delta\text{-}chr}$ stand.

- In this talk we introduce results explaining

$$\mathfrak{Refl}_{>\omega\text{-}col} \leq \mathfrak{Refl}_{>\omega\text{-}chr}$$

holds and, in certain cases, the corresponding inequality also holds for regular cardinals δ with $\delta^{<\delta} = \delta$.

- **Spoiler:**

$$\mathfrak{Refl}_{>\omega\text{-}col} \leq \mathfrak{Refl}_{\omega\text{-}CC\downarrow} \leq \mathfrak{Refl}_{\omega\text{-}Rado} \leq \mathfrak{Refl}_{\omega\text{-}Galvin} \leq \mathfrak{Refl}_{>\omega\text{-}chr}$$

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Reflection cardinal for FRP (1/3)

- For a regular cardinal δ and a cardinal $\lambda > \delta$, let $E^\lambda_\delta = \{\alpha \in \lambda : \mathrm{cf}(\alpha) = \delta\}$.
- For a regular cardinal $\delta \geq \omega$, the reflection cardinal for δ-Fodor-type Reflection Principle is defined as follows:

FRP$(\delta, {<}\kappa, \lambda)$: For any stationary $S \subseteq E^\lambda_\delta$ and $g : S \to [\lambda]^\delta$ s.t. $g(\alpha) \subseteq \alpha$ for $\alpha \in S$, there is $\alpha^* < \lambda$ s.t. $\delta < \mathrm{cf}(\alpha^*) < \kappa$ and $\{x \in [\alpha^*]^\delta : \sup(x) \in S, g(\sup(x)) \subseteq x\}$ is stationary in $[\alpha^*]^\delta$

▷ $\mathfrak{Refl}_{\delta\text{-}FRP} = \min\{\kappa :$ FRP$(\delta, {<}\kappa, \lambda)$ for all regular $\lambda > \delta$ holds.$\}$

- The Fodor-type Reflection Principle (FRP) is defined by:

$$\text{FRP} \;\Leftrightarrow\; \mathfrak{Refl}_{\omega\text{-}FRP} = \aleph_2$$

Reflection cardinal for FRP (2/3)

Theorem 5 (H. Sakai and S.F. (2012))

Suppose that δ is regular and $\kappa \geq \mathfrak{Refl}_{\delta\text{-}FRP}$ holds. Then, for any graph $G = \langle G, K \rangle$, if $col(G \restriction I) \leq \delta$ holds for all $I \in [G]^{<\kappa}$ then $col(G) \leq \delta$.

A Sketch of Proof: By induction on the cardinality λ of the graph $G = \langle G, K \rangle$.

- If λ is singular Shelah's Singular Compactness Theorem will do.
- For regular λ the following lemma is used:

For $I \subseteq G$ and $p \in G$, let $K_I(p) = \{q \in I \,:\, p\, K\, q\}$.

Lemma 6 (Erdős, Hajnal (1966))

If $\langle G_\alpha \,:\, \alpha < \mu \rangle$ is a filtration of G s.t. $col(G_\alpha) \leq \delta$ and $|\,K_{G_\alpha}(p)\,| < \delta$ for all $\alpha < \mu$ and $p \in G_{\alpha+1}$. Then we have $col(G) \leq \delta$.

$\square$ (Theorem 5)

Reflection cardinal for FRP (3/3)

Theorem 5 (H. Sakai and S.F. (2012))

Suppose that δ is regular and $\kappa \geq \mathfrak{Refl}_{\delta\text{-}FRP}$ holds. Then, for any graph $G = \langle G, K \rangle$, if $col(G \restriction I) \leq \delta$ holds for all $I \in [G]^{<\kappa}$ then $col(G) \leq \delta$.

Corollary 7

$\mathfrak{Refl}_{>\omega\text{-}col} \leq \mathfrak{Refl}_{\omega\text{-}FRP}$.

Theorem 8 (T. Usuba)

$\mathfrak{Refl}_{>\omega\text{-}col} = \mathfrak{Refl}_{\omega\text{-}FRP}$.

Corollary 9

FRP *is equivalent to* $\mathfrak{Refl}_{>\omega\text{-}col} = \aleph_2$.

Problem 2. Does Usuba's Theorem hold for $\delta > \omega$?

A version of Chang's conjecture (1/2)

- For a sufficiently large (relative to λ) regular θ, let $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \sqsubset \rangle$ where $\sqsubset$ is a well-ordering on $\mathcal{H}(\theta)$. For regular δ with $\delta^{<\delta} = \delta$, let

$\mathsf{CC}^{\downarrow}(\delta, <\kappa, \lambda)$: For any $M \prec \mathcal{M}$ with $|M| = \delta$, $[M]^{<\delta} \subseteq M$, δ, κ, $\lambda \in M$ and $\delta \subseteq M$; and for any $\alpha \in \lambda$ there is $M^* \prec \mathcal{M}$ and $\alpha^* \in \lambda \setminus \alpha$ s.t. $M \prec M^*$, $\delta < \mathrm{cf}(\alpha^*) < \kappa$ and $\alpha^* = \min(\lambda \cap M^* \setminus \sup(\lambda \cap M))$.

- $\mathfrak{Refl}_{\delta\text{-}\mathsf{CC}^{\downarrow}} = \min\{\kappa \in \mathrm{Card} : \delta^+ < \kappa,\ \mathsf{CC}^{\downarrow}(\delta, <\kappa, \lambda)$ holds for all $\lambda \geq \kappa\}$

A version of Chang's conjecture (2/2)

Lemma 10

Suppose that δ is a regular cardinal with $\delta^{<\delta} = \delta$, $\delta^+ < \kappa$ a cardinal and λ is a regular cardinal with $\mu^\delta < \lambda$ for all $\mu < \lambda$. Then $\mathsf{CC}^{\downarrow}(\delta, <\kappa, \lambda)$ implies $\mathsf{FRP}(\delta, <\kappa, \lambda)$.

The Idea of the Proof. Use α^* in $\mathsf{CC}^{\downarrow}(\delta, <\kappa, \lambda)$ as the α^* in $\mathsf{FRP}(\delta, <\kappa, \lambda)$. □(Lemma 10)

Corollary 11

$\mathfrak{Refl}_{>\omega\text{-}col} \leq \mathfrak{Refl}_{\omega\text{-}CC^{\downarrow}}$.

Proof. By Lemma 10 and (the proof of) Theorem 5.

□(Corollary 11)

Rado Conjecture and $\mathrm{CC}^{\downarrow}$ (1/2)

- The reflection cardinal for Rado's Conjecture is defined as follows

$\mathrm{RC}(\delta, {<}\kappa, \lambda)$: For any tree of cardinality λ if T is not δ-special then there is a $T' \in \mathcal{S}_{<\kappa}(T)$ which is not δ-special.

- $\mathfrak{Refl}_{\delta\text{-}Rado} = \min\{\kappa : \mathrm{RC}(\delta, {<}\kappa, \lambda) \text{ holds for all } \lambda \geq \kappa\}$.

- In the notation at the begining of this talk, $\mathfrak{Refl}_{\delta\text{-}Rado}$ is $\mathfrak{Refl}(\mathcal{C}, \mathcal{P})$ where $\mathcal{C}$ is trees and $\mathcal{P}$ is the property "not δ-special".

- Rado's Conjecture (RC) is the assertion $\mathfrak{Refl}_{\omega\text{-}Rado} = \aleph_2$.

- Rado's Conjecture can be forced starting from a model with a strongly compact cardinal κ and Levi-collapse cardinals $< \kappa$ by countable conditions.

Rado Conjecture and $\mathsf{CC}^{\downarrow}$ (2/2)

Theorem 12

Suppose that δ is a regular cardinal with $\delta^{<\delta} = \delta$ then $\mathsf{RC}(\delta, <\kappa, \lambda^{\delta})$ implies $\mathsf{CC}^{\downarrow}(\delta, <\kappa, \lambda)$.

Sketch of the Proof. Assume $\mathsf{CC}^{\downarrow}(\delta, <\kappa, \lambda)$ does not hold. Then we can construct a tree T consisting of $\in$-chain of elementary submodels of $\mathcal{M}$ of cardinality δ s.t. T witnesses the negation of $\mathsf{RC}(\delta, <\kappa, \lambda^{\delta})$. $\square$(Theorem 12)

Corollary 13

For a regular cardinal δ with $\delta^{<\delta} = \delta$, we have $\mathfrak{Refl}_{\delta\text{-}CC^{\downarrow}} \leq \mathfrak{Refl}_{\delta\text{-}RC}$.

Corollary 14

RC implies FRP.

Reflection cardinal for Galvin's Conjecture (1/2)

- The reflection cardinal of Galvin's Conjecture can be formulated as follows:

  - $\mathfrak{Refl}_{\delta\text{-}Galvin} = \min\{\kappa :$ For any partial ordering P, if P is not the union of less than or equal to δ many linear subsets, then there is a subordering P' of P of cardinality $< \kappa$ s.t. P' is not the union of less than or equal to δ many linear subsets$\}$

- Galvin's Conjecture is the statement $\mathfrak{Refl}_{\omega\text{-}Galvin} = \aleph_2$.

  - The consistency of Galvin's Conjecture is a long-standing open problem.

Reflection cardinal for Galvin's Conjecture (2/2)

Theorem 15 (S. Todorcevic (2011))

For any infinite cardinal δ we have $\mathfrak{Refl}_{\delta\text{-}Rado} \leq \mathfrak{Refl}_{\delta\text{-}Galvin} \leq \mathfrak{Refl}_{>\delta\text{-}chr}$.

Proof.

- The first inequality: Suppose that $T = \langle T, \leq_T \rangle$ is a tree witnessing $\kappa < \mathfrak{Refl}_{\delta\text{-}Rado}$. Let $\lhd$ be a well-ordering on T and Let $\lhd_T$ be the ordering on T defined by $t \lhd_T t' \;\Leftrightarrow\; t$ and t' are incomparable in T and first branching nodes t_0 and t'_0 below t and t' respectively are s.t. $t_0 \lhd t'_0$. $\langle T, \lhd_T \rangle$ is then a partial ordering witnessing $\kappa < \mathfrak{Refl}_{\delta\text{-}Galvin}$.
- The second inequality: Suppose that $P = \langle P, \leq_P \rangle$ is a partial ordering witnessing $\kappa < \mathfrak{Refl}_{\delta\text{-}Galvin}$. Let K be the binary relation on P defined by $\langle p, q \rangle \in K \;\Leftrightarrow\; p$ and q are incomparable w.r.t. $\leq_P$. $\langle P, K \rangle$ is then a graph witnessing $\kappa < \mathfrak{Refl}_{>\delta\text{-}chr}$.

□(Theorem 15)

Summary and applications (1/3)

- The inequalities we obtained sofar can be put together as the following:

$$\begin{aligned}\mathfrak{Refl}_{>\omega\text{-}col} = \mathfrak{Refl}_{\omega\text{-}FRP} \leq \mathfrak{Refl}_{\omega\text{-}Rado} \\ \leq \mathfrak{Refl}_{\omega\text{-}Galvin} \leq \mathfrak{Refl}_{>\omega\text{-}chr}\end{aligned}$$

Summary and applications (2/3)

Theorem 16

For a regular cardinals $\delta < \lambda$, if there is a non reflecting stationary subset of E^{λ}_{δ}, then there is a graph $G = \langle G, K \rangle$ s.t. (*) *$col(G \restriction I) \leq \delta$ for all $I \in [G]^{<\lambda}$ but* (**)*$col(G) > \delta$.*

Proof. Let $S \subseteq E^{\lambda}_{\delta}$ be a non-reflecting stationary set and let $\langle c_\alpha : \alpha \in S \rangle$ a ladder system on S ($c_\alpha \subseteq \alpha \setminus Limits$ is cofinal in α and $\mathrm{ot}(c_\alpha) = \delta$).

Then, letting,

$K = \{\langle \alpha, \beta \rangle, \langle \beta, \alpha \rangle : \alpha \in S, \beta \in c_\alpha\}$,

$G = \langle \lambda, K \rangle$ is as desired (Apply Lemma 6 to show (*)).

□(Theorem 16)

Summary and applications (3/3)

Corollary 17 (Shelah, SH1006)

If there is a non reflecting stationary subset of E^{λ}_{δ}, then there is a graph $G = \langle G, K\rangle$ of size λ^{δ} s.t. $(*)$ $\mathrm{chr}(G \restriction I) \leq \delta$ for all $I \in [G]^{<\lambda}$ but $(**)\mathrm{chr}(G) > \delta$.

Proof. By Theorem 16 and (the constructions in the proofs of) the inequalities. □(Corollary 17)


Děkuji vám za pozornost !

Coloring number of a graph

- A graph $G = \langle G, K \rangle$ has the coloring number $\leq \delta \in \mathrm{Card}$ if there is a well-ordering $\sqsubseteq$ on G s.t. for all $p \in G$ the set

$$\{q \in G \,:\, q \sqsubseteq p \text{ and } q \, K \, p\}$$

  has cardinality $< \delta$. Such a well-ordering can be always chosen such that it has the order type of the cardinality of G.

- The coloring number $col(G)$ of a graph G is the minimal cardinal among such δ as above.

δ-Special Tree

- For a cardinal δ, a tree T is said to be δ-special if T can be represented as the union of δ-many pairwise incomparable sets (antichains).

- If T is δ-special then there is no δ^+-branch in T.

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