Games and perfect independent subsets of the generalized Baire space

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The generalized Baire space

Let κ be an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$.

The domain of the κ-Baire space is the set ${}^{\kappa}\kappa$ of functions $f : \kappa \to \kappa$. Its topology is given by the basic open sets

$$N_p = \{f \in {}^{\kappa}\kappa : p \subseteq f\},$$

where $p \in {}^{<\kappa}\kappa$ (i.e., $p : \alpha \to \kappa$ for some $\alpha < \kappa$).

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κ-Borel sets: close the family of open subsets under intersections and unions of size $\leq \kappa$ and complementation.

κ-perfect sets

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$X \subseteq {}^{\kappa}\kappa$ is a *κ-perfect set* if $X = [T]$ for some κ-perfect tree T.

A game characterizing κ-perfectness

Definition (Väänänen)

Let $X \subseteq {}^\kappa\kappa$. Then $G_\kappa(X)$ is the following game.

I	n_0		n_1		$\dots$		n_α		$\dots$	
II		x_0		x_1		$\dots$		x_α		$\dots$

I plays $n_\alpha < \kappa$ such that $n_\alpha > n_\beta$ for all $\beta < \alpha$, and $n_\alpha = \sup_{\beta<\alpha} n_\beta$ at limits α.

II responds with $x_\alpha \in X$ such that $x_\alpha \restriction n_{\beta+1} = x_\beta \restriction n_{\beta+1}$ but $x_\alpha \neq x_\beta$ for all $\beta < \alpha$.

Player II wins, if she can make all her κ moves.

- A closed set X contains a κ-perfect subset iff II wins $G_\kappa(X)$.
- When $X \subseteq {}^\kappa\kappa$ is arbitrary,
  II wins $G_\kappa(X)$ iff there exists $Y \subseteq X$ such that $\overline{Y}$ is κ-perfect,

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- When $X \subseteq {}^{\kappa}\kappa$ is arbitrary,
  II wins $G_\kappa(X)$ iff there exists $Y \subseteq X$ such that $\overline{Y}$ is κ-perfect,
- X is κ-*scattered* iff Player I wins $G_\kappa(X)$.

A dichotomy about $G_\kappa(X)$ from a weakly compact cardinal

For all $X \subseteq {}^\kappa\kappa$,

(1) either $|X| \leq \kappa$ or Player II wins $G_\kappa(X)$ (i.e. there is $Y \subseteq X$ such that $\overline{Y}$ is κ-perfect).

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Theorem (Schlicht, Sz.)

If $\lambda > \kappa$ is weakly compact, then the Lévy-collapse Col$(\kappa, < \lambda)$ forces that:

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- If $\lambda > \kappa$ is inaccessible, then $\mathrm{Col}(\kappa, <\lambda)$ forces that (1) holds for closed subsets of ${}^\kappa\kappa$, and even subsets of ${}^\kappa\kappa$ definable from ordinals and subsets of κ (Schlicht).

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- If $\lambda > \kappa$ is inaccessible, then $\mathrm{Col}(\kappa, < \lambda)$ forces that (1) holds for closed subsets of ${}^\kappa\kappa$, and even subsets of ${}^\kappa\kappa$ definable from ordinals and subsets of κ (Schlicht).
- It was known that if $\lambda > \kappa$ is measurable, then $\mathrm{Col}(\kappa, < \lambda)$ forces that (1) for all subsets of ${}^\kappa\kappa$ (Galvin, Jech, Magidor; Väänänen).

A dichotomy for infinitely many $\mathbf{\Sigma}^0_2(\kappa)$ relations

R is a $\mathbf{\Sigma}^0_2(\kappa)$ *relation* on a topological space X iff

R is a union of $\leq \kappa$ many closed subsets of ${}^k X$ for some $1 \leq k < \omega$.

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Let $\mathcal{R}$ be a collection of finitary relations on X.

$Y \subseteq X$ is $\mathcal{R}$-*independent* if for all $1 \leq k < \omega$ and k-ary $R \in \mathcal{R}$ we have:
$(x_1, \dots, x_k) \notin R$ for all pairwise distinct $x_1, \dots, x_k \in Y$.

Proposition (Sz.)

Assume $\diamondsuit_\kappa$ or κ is inaccessible.

Let $\mathcal{R}$ be a collection of $\leq \kappa$ many $\mathbf{\Sigma}^0_2(\kappa)$ relations on ${}^\kappa\kappa$.

If II wins $G_\kappa(Y)$ for some $\mathcal{R}$-independent $Y \subseteq {}^\kappa\kappa$, then there exists a κ-perfect $\mathcal{R}$-independent subset of ${}^\kappa\kappa$.

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Corollary

If $\lambda > \kappa$ is weakly compact, then in $V^{\mathsf{Col}(\kappa,<\lambda)}$ the following holds:

Let $\mathcal{R}$ be a collection of $\leq \kappa$ many $\mathbf{\Sigma}^0_2(\kappa)$ relations on $X = {}^\kappa\kappa$
(or even on a κ-analytic subset $X \subseteq {}^\kappa\kappa$).

If there is an $\mathcal{R}$-independent $Y \subseteq X$ of size $> \kappa$, then
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- Countable version of this dichotomy: Kubiś (2003), Doležal, Kubiś (2015).

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there exists a κ-perfect $\mathcal{R}$-independent subset of X.

- This was known for $\lambda > \kappa$ measurable (Sz., Väänänen).
- The dichotomy in the corollary implies that κ^+ is inaccessible in L.

A version that does not need large cardinals

Theorem (Sz.)

Assume $\diamondsuit_\kappa$ or κ is inaccessible.

Let $\mathcal{R}$ be a collection of $\leq \kappa$ many $\mathbf{\Sigma}^0_2(\kappa)$ relations on ${}^\kappa\kappa$.

If a κ-version of the statement "there exist $\mathcal{R}$-independent subsets of arbitrarily large Cantor-Bendixson rank" holds,

then there exists a κ-perfect $\mathcal{R}$-independent subset of ${}^\kappa\kappa$.

- Countable version of this dichotomy: Kubiś (2003),
  Doležal, Kubiś (2015).

Trees as "Cantor-Bendixson ranks" for the κ-Baire space

Definition (Väänänen)

Let $X \subseteq {}^\kappa\kappa$, and let T be any tree. $G_T(X)$ is the following game.

I	t_0, n_0		t_1, n_1		$\dots$		t_α, n_α		$\dots$
II		x_0		x_1		$\dots$		x_α		$\dots$

I plays $t_\alpha \in T$ and $n_\alpha < \kappa$ such that $t_\alpha >_T t_\beta$ and $n_\alpha > n_\beta$ for all $\beta < \alpha$, and $n_\alpha = \sup_{\beta<\alpha} n_\beta$ at limits α.

II responds with $x_\alpha \in X$ such that $x_\alpha \restriction n_{\beta+1} = x_\beta \restriction n_{\beta+1}$ but $x_\alpha \neq x_\beta$ for all $\beta < \alpha$.

The first player who can not move loses, and the other player wins.

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- If T consists of just one branch of length κ, then $G_T(X)$ is same game as $G_\kappa(X)$.

For an ordinal α, let

$B_\alpha =$ tree of descending sequences of elements of α.

Claim

The Cantor-Bendixson rank of X is $\geq \alpha$ (i.e. $X^{(\alpha)} \neq \emptyset$)

$$\textit{iff Player } \mathrm{I} \textit{ wins } G_{B_\alpha}(X)$$

$$\textit{iff Player } \mathrm{II} \textit{ does not win } G_{B_\alpha}(X).$$

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Claim

The Cantor-Bendixson rank of X is $\geq \alpha$ (i.e. $X^{(\alpha)} \neq \emptyset$)

iff Player I wins $G_{B_\alpha}(X)$

iff Player II does not win $G_{B_\alpha}(X)$.

Two ways to generalize Cantor-Bendixson ranks for $X \subseteq {}^\kappa\kappa$ using trees T without κ-branches:

"X is simple iff Player I wins $G_T(X)$"

or

"X is simple iff Player II does not win $G_T(X)$."

Recall: II wins $G_\kappa(X)$ iff X has a subset whose closure is κ-perfect.

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Theorem (Sz.)

Assume $\diamondsuit_\kappa$ or κ is inaccessible.

Let $\mathcal{R}$ be a collection of $\leq \kappa$ many $\mathbf{\Sigma}_2^0(\kappa)$ relations on ${}^\kappa\kappa$.

Then either

- *there exists a κ-perfect $\mathcal{R}$-independent subset of ${}^\kappa\kappa$, or*
- *there exists a tree T without κ-branches, $|T| \leq 2^\kappa$,*

  such that

  Player II does not win $G_T(X)$ for any $\mathcal{R}$-independent $X \subseteq {}^\kappa\kappa$.

When κ is inaccessible, we can have $|T| \leq \kappa$.

Thank you for your attention!