A filter on a collection of finite sets. A non-bisequential Eberlein compactum.

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A space X is *bisequential* [resp. *biradial*] if, whenever p is a point of X and $\mathcal{U}$ is an ultrafilter converging to p, then there is a countable [*resp.* totally ordered] $B \subset \mathcal{U}$ such that the filter generated by B converges to p. A compact space is called *Eberlein compact* if it can be embedded in a Banach space with the weak topology.

Of course, every bisequential space is biradial.

Example. Let X be the set of all antichains in $\omega_1 \times \omega_1$, with the product topology. Then X is Eberlein compact and not bisequential, but is biradial, while $X \times \omega + 1$ is Eberlein compact, but not biradial.

In contrast, every uniform Eberlein compact space (=weakly compact subset of a Hilbert space) of size smaller than the first uncountable measurable cardinal is bisequential.

There are also examples of Eberlein compacta which are not uniformly Eberlein compact, but still bisequential. However, the following problem is still unsolved: is there a Banach space of which every weakly compact subset is bisequential, but not every weakly compact subset is uniform Eberlein compact?

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Why would one look for an example of such space?

Eberlein compacta are Frechet-Urysohn, i.e. for every subset A and every $x \in \overline{A}$ there exists a sequence $x_1, x_2, \ldots$ of elements of A converging to x.

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Let $\mathcal{X}$ be the set of the graphs of all strictly decreasing functions $f\colon S \to \omega_1$, where $S \subset \omega_1$.

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Since every set can be identified with its characteristic function, we can view $\mathcal{X}$ as a subset of $\{0,1\}^{\omega_1 \times \omega_1}$. We endow $\mathcal{X}$ with the topology of pointwise convergence, i.e. the topology inherited from the usual product topology on $\{0,1\}^{\omega_1 \times \omega_1}$.

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A compact space is called an Eberlein compactum if it is homeomorphic to a subspace of some Banach space in its weak topology.

Theorem (folklore)

Let X be a set and let $\mathcal{A}$ be a family consisting of some finite subsets of X. If $K = \{\chi_A \colon A \in \mathcal{A}\}$ is a closed subspace of $\{0,1\}^X$ then K is an Eberlein compactum.

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Lemma

For a subset A of $\omega_1 \times \omega_1$, we will denote the vertical and horizontal sections of A at α by

$$A_\alpha = \{\beta\colon (\alpha, \beta) \in A\}, \qquad A^\beta = \{\alpha\colon (\alpha, \beta) \in A\}.$$

Let $\mathcal{I}$ be σ-ideal consisting of sets $A \subset \omega_1 \times \omega_1$ such that for all but countably many α, $|A_\alpha| \leq \aleph_0$ and $|A^\alpha| \leq \aleph_0$.

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Proof of lemma

Since $A \in \mathcal{I}$, there exists an ordinal $\alpha < \omega_1$ such that

$$\forall \alpha < \beta < \omega_1 \ \ |A_\beta| < \aleph_1 \wedge |A^\beta| < \aleph_1.$$

$B_1, \ldots, B_n \in \mathcal{P}(\omega_1 \times \omega_1) \setminus \mathcal{I}$, therefore for any $i = 1, \ldots, n$ there exist uncountably many ordinals $\beta < \omega_1$ such that

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We can inductively define ordinals $\beta_1, \beta_2, \ldots, \beta_k$ such that:

- $\alpha < \beta_1 < \beta_2 < \ldots < \beta_k$,
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Analogously, there exist ordinals $\gamma_{k+1}, \gamma_{k+2}, \ldots, \gamma_n$ such that

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It follows that $(\beta_i, \gamma_i) \in B_i$ and $(\beta_i, \gamma_i) \notin A$ for any $1 \leq i \leq n$. Moreover

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The filter

For any $A, B_1, \ldots, B_n \subset \omega_1 \times \omega_1$ define

$$\mathcal{F}_A = \{C \in \mathcal{X} : A \cap C = \emptyset\}$$

and

$$\mathcal{F}_A(B_1, B_2, \ldots, B_n) = \{C \in \mathcal{X} : A \cap C = \emptyset \wedge \forall i \leq n \quad B_i \cap C \neq \emptyset\}.$$

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For any $A, A' \in \mathcal{I}$ and $B_1, \ldots, B_n, B'_1, \ldots, B'_m \in \mathcal{P}(\omega_1 \times \omega_1) \setminus \mathcal{I}$ we have $A \cup A' \in \mathcal{I}$ and

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Bisequential spaces

A sequence of non-empty subsets $A_0, A_1, \ldots$ of a topological space X converges to a point $x \in X$ if for any neighbourhood U of x there exists an integer n_0 such that for any $n > n_0$ we have $A_n \subset U$.

An ultrafilter $\mathcal{U} \subset \mathcal{P}(X)$ is convergent to $x \in X$ if every neighbourhood U of x is an element of $\mathcal{U}$.

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Proof

Let $\mathscr{U}$ be any ultrafilter extending the filter $\mathscr{F}$. For the sake of contradiction assume that $\mathcal{X}$ is bisequential. Then there exists a decreasing sequence $\mathcal{U}_0, \mathcal{U}_1, \ldots$ of elements of $\mathscr{U}$ such that for any basic neighbourhood $\mathcal{F}_A$ of $\emptyset$ there exists a positive integer i such that $\mathcal{U}_i \subset \mathcal{F}_A$.

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$$\mathcal{U}_i \subset \bigcap_{(\alpha,\beta)\in A_i} \mathcal{F}_{\{(\alpha,\beta)\}} = \mathcal{F}_{A_i}.$$

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$$A_i = \left\{(\alpha, \beta) \in \omega_1 \times \omega_1 \colon \mathcal{U}_i \subset \mathcal{F}_{\{(\alpha,\beta)\}}\right\}.$$

Then $\bigcup_{i\in\omega} A_i = \omega_1 \times \omega_1$. Moreover

$$\mathcal{U}_i \subset \bigcap_{(\alpha,\beta)\in A_i} \mathcal{F}_{\{(\alpha,\beta)\}} = \mathcal{F}_{A_i}.$$

Thus $\mathcal{F}_{A_i} \in \mathscr{U}$.

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The space $\mathcal{X}$ was studied before Nyikos' announcement, it appeared in a paper[1] by Leiderman and Sokolov as an example of an Eberlein compactum which is not uniform Eberlein compactum.

We can use the non-bisequentiality of $\mathcal{X}$ (and other well-known results) to prove that $\mathcal{X}$ is not uniform Eberlein compactum. This gives a new proof of this fact.

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